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ON MACH REFLECTION AND STRENGTH OF REFLECTED SHOCK

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WRIGHT AIR DEVELOPMENT CENTER

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FOREWORD

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ABSTRACT

The configuration of non-stationary Mach reflection has a conical similarity in time-space, since there is no fundamental time or space interval involved. This property is easily shown (Part Ib), and can be used to simplify the determination of various disturbance quantities.

For this type of reflection the strength of the reflected shock depends on that of the incident shock and also on the deflection angle. To the first approximation, this reflected wave is a sonic front. Thus, the resulting boundary-value problem in linearized theory is relatively simple, and has been attacked by several investigators[1,2,3]. The extension to second-order theory is discussed (Part Id) but is not carried out in detail.

Based on the results obtained from linear theory, it is shown how the shock strength can actually be determined to the second order (Part II). The strength of this reflected shock is found to be of second order, and it vanishes at the triple point.

PUBLICATION REVIEW

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I. Mach Reflection

(a) General Consideration

Let us consider a plane shock I propagating into still air with a constant velocity U_s in the positive x direction; the air behind the shock has a uniform velocity U_1 . Let us further denote the flow region ahead of the shock by 0 , and that behind by 1 .

With respect to a frame attached to shock I , this is a stationary flow field. The strength of shock I is measured by the pressure ratio $\xi = p_1 / p_0$, or by the Mach number in flow 1 referred to frame I , i.e.,

$$\sigma = \frac{U_s - U_1}{a_1}$$

Since

$$\frac{1}{\xi} = \frac{2\gamma \sigma^2 - (\gamma - 1)}{\gamma + 1} , \qquad 1 - \sigma^2 = \frac{\gamma + 1}{2\gamma}\left(1 - \frac{1}{\xi}\right)$$

$$U_1 = \frac{2(1 - \sigma^2)}{(\gamma + 1)\sigma}\, a_1 , \qquad U_s = \frac{2 + (\gamma - 1)\sigma^2}{(\gamma + 1)\sigma}\, a_1$$

two immediate conclusions can be drawn.

i) Since $\xi > 1$, i.e., $\sigma < 1$, with respect to flow in 1 , shock I always propagates with subsonic speed, so that any disturbance in 1 catches up with the shock. Thus when the shock hits a wall corner, the appearance of Mach reflection is natural unless the turning angle δ be so large that the so-called "regular reflection" occurs.
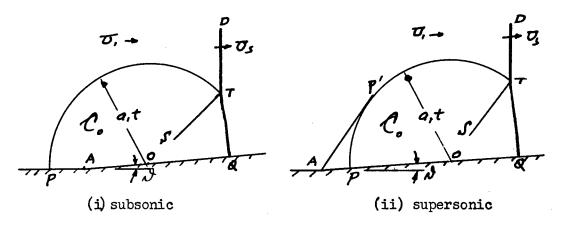
ii) The Mach number of flow $/$ is given by

$$M_1 = \frac{U_1}{a_1} = \frac{2(1 - \sigma^2)}{(\gamma+1)\,\sigma}$$

Thus the flow behind shock is supersonic $(M_1 > 1)$ when

$$\sigma < \sqrt{(\frac{\gamma+1}{4})^2 - 1} - \frac{\gamma+1}{4} = 0.566$$

or

$$\xi > \frac{2.4}{2.8\,\sigma^2 - .4} = \frac{2.4}{.496} = 4.84$$

and subsonic $(M_1 < 1)$ when $\sigma > 0.566$, or $\xi < 4.84$.

When such a propagating plane shock hits a corner, the configuration of resulting Mach reflection can be stated as follows: There is a triple point T ; part PT of shock I remains straight, being unaffected by the signal from corner A ; part $\widehat{TQ}$ is curved, due to effect of deflection by A ; $\widehat{TP}$ is the reflected wave signaling the presence of A (in the first approximation, the entire front is sonic in subsonic case, while only part $\widehat{TP'}$ is sonic in supersonic case); and a slip stream TS appears, across which entropy and density (but not pressure) are discontinuous. (Fig. 1, 2). The whole region C_o propagates to the right with velocity U_1 and at the same time expands radially about origin O with speed of sound a_1 . Inside region C_o the flow is non-steady. However, by a proper transformation of coordinates it can be shown that there exists a conical property, with O as apex, and time t as axis of the cone, so that the problem can be reduced to a two dimensional steady one. The flow will then be signified by two parameters in its conical plane: strength of the incident shock, and angle of wall corner.

(i) subsonic   (ii) supersonic

An exact solution to the problem is mathematically not practical, so a linear solution is first sought. This has been done by different authors under different assumptions.

If ϑ is very small, reflected wave is weak. Vorticity Ω above the slipstream can be neglected (indeed, $\Omega < O(\vartheta^3)$). However, unless incident shock is also weak, vorticity in region OQT is by no means negligible.

(1)

By assuming a weak shock and small corner angle, Bargmann investigated the disturbance-velocity distribution in the whole region C_o by introducing a velocity potential. This, however, necessitates an assumption that the shock be so weak that the existence of the slip stream and the region of rotational flow can be neglected. Later, not assuming weak shock, Lighthill [2] showed that the disturbance pressure p in whole region C_o satisfies one single conical equation in spite of the presence of the slip stream and the rotational flow, and so he calculated the pressure distribution along both the wall and Mach shock. Recently, Ludloff [3], using the Lorentz transformation, determined both pressure and density field in C_o. His analysis showed the slip stream lies along the radial line OT, as expected, flow in region OTQ being rotational, while that in region OTP irrotational.

Utilizing the results of previous authors, it should be possible to determine the disturbance velocity u, v in region OTP, taking into consideration the slip stream, so that the incident shock need not be weak, although the deflection angle would still be assumed small. This solution should prove an immediate improvement over Bargmann's.

Although we allow the shock strength to be not small, we shall in the present report still limit it so that the flow behind the shock is subsonic; the case where the after-flow is supersonic will be left to further investigation.

Essentially, the procedure followed will consist in taking the uniform flow behind the straight shock as our basic flow, and deflection angle ϑ as a perturbation. Thus all the perturbing quantities will be expressed in orders of ϑ. Based on the results of the present investigation, further approximations can be carried out using a technique suggested by Lighthill.

(b) Conical Similarity

Designate the uniform flow field behind the propagating straight shock by 1, the disturbed flow behind the curved Mach shock and inside the reflected front R by 2. Let velocity, pressure, density and entropy be denoted by q_i, p_i, ρ_i, s_i ($i = 1, 2$), and the corresponding non-dimensional perturbed quantities by (u, v), p, ρ, s, so that

$$q_2 = q_1 (1+u, v)$$
$$p_2 = p_1 + q_1 q_1 \rho_1 \bar{p}$$
$$\rho_2 = \rho_1 (1 + \bar{\rho})$$
$$M_1 = q_1 / a_1$$

Then referred to the fixed coordinate system with corner as origin, the equation of continuity, equations of motion, and equation of energy (isentropy along flow line) take on the following forms

$$\frac{\partial \rho_2}{\partial t} + q_1 (1+u) \frac{\partial \rho_1}{\partial X} + q_1 v \frac{\partial \rho_1}{\partial Y} + \rho_2 q_1 \left(\frac{\partial u}{\partial X} + \frac{\partial v}{\partial Y} \right) = 0$$

$$\frac{\partial u}{\partial t} + q_1 (1+u) \frac{\partial u}{\partial X} + q_1 v \frac{\partial u}{\partial Y} + \frac{q_1 \rho_1}{\rho_2} \frac{\partial \bar{p}}{\partial X} = 0 \qquad (1)$$

$$\frac{\partial v}{\partial t} + q_1 (1+u) \frac{\partial v}{\partial X} + q_1 v \frac{\partial v}{\partial Y} + \frac{q_1 \rho_1}{\rho_2} \frac{\partial \bar{p}}{\partial Y} = 0$$

$$\frac{\partial S_2}{\partial t} + q_1 (1+u) \frac{\partial S_2}{\partial X} + q_1 v \frac{\partial S_2}{\partial Y} = 0$$

On introducing the transformation

$$x = \frac{X - q_1 t}{a_1 t} \qquad \qquad y = \frac{Y}{a_1 t} \qquad (2)$$

we are referring the flow to a new reference frame which has its origin at o and moves together with flow, and we are contracting the whole flow field by a scale $a_1 t$. This means that, at any instant, the sonic front that originated from o in the physical plane

$$(X - q_1 t)^2 + Y^2 = (a_1 t)^2 \qquad (3)$$

reduces to the unit circle in the transformed plane:

$$x^2 + y^2 = 1 \qquad (4)$$

If by this transformation we can eliminate the time dependence, then our problem simplifies to a two-dimensional steady one. That is, the non-steady motion will have a conical property in space-time; the apex of the cone being at o , the axes of the cone being the time ordinate, normal to the plane of the motion.

Now since
$$\frac{\partial}{\partial \bar{x}} = \frac{1}{q_1 t}\frac{\partial}{\partial x} \qquad \frac{\partial}{\partial \bar{y}} = \frac{1}{q_1 t}\frac{\partial}{\partial y}$$
$$\frac{\partial}{\partial t} = -\left(\frac{x}{t} + \frac{q_1}{q_1 t}\right)\frac{\partial}{\partial x} - \frac{y}{t}\frac{\partial}{\partial y}$$

we have, on introducing the transformation, the following system of

equations of continuity, motion and energy

$$D\rho = M_1(1+\rho)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$D u = \frac{1}{1+\rho}\frac{\partial p}{\partial x}$$

$$D v = \frac{1}{1+\rho}\frac{\partial p}{\partial y}$$ (5)

$$D p = (1 + \gamma M_1 p)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

where
$$D = -t\left[\frac{\partial}{\partial t} + q_1(1+u)\frac{\partial}{\partial x} + q_1 v\frac{\partial}{\partial y}\right]$$

$$= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - M_1\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right).$$

The form of the system of equations clearly shows that the motion is

indeed conical in time-space.

Using vector notation $\underline{q} = (u, v)$, $\underline{r} = (x, y)$, these take

the form

$$(\underline{r} - M_1\underline{q})\cdot\nabla\underline{q} = \frac{1}{1+\rho}\nabla p$$

$$(\underline{r} - M_1\underline{q})\cdot\nabla\rho = M_1(1+\rho)\nabla\cdot\underline{q}$$

$$(\underline{r} - M_1\underline{q})\cdot\nabla p = (1 + \gamma M_1 p)\nabla\cdot\underline{q}$$ (6)

(∇ refers to $\underline{r}$ plane hereafter)

(c) Linearization

Let us suppose that all the perturbation quantities and their

derivatives are small, i.e.

$$u, v, p, \rho, \text{etc.} \ll 1$$

Then we have

$$\underline{r} \cdot \nabla \rho = M_1 \nabla \cdot \vec{q}$$

$$\underline{r} \cdot \nabla \vec{q} = \nabla p \qquad (7)$$

$$\underline{r} \cdot \nabla p = \nabla \cdot \vec{q}$$

or

$$x \frac{\partial \rho}{\partial x} + y \frac{\partial \rho}{\partial y} = M_1 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{\partial p}{\partial x}$$

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = \frac{\partial p}{\partial y}$$

$$x \frac{\partial p}{\partial x} + y \frac{\partial p}{\partial y} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \qquad (7')$$

In conformity with the approximation, it may be noted that the reflected front $\widehat{PT}$ coincides with the sonic front, i.e., $r = 1$, and slip stream ST coincides with radial line $\theta = \cos^{-1} (M_s - M_1)$. By eliminating the undesired variables, it is easily obtained.

i) Pressure perturbation

$$\nabla \cdot \underline{r} \cdot \nabla \vec{q} = (\underline{r} \cdot \nabla + 1) \nabla \cdot \vec{q} = (\underline{r} \cdot \nabla + 1)(\underline{r} \cdot \nabla p)$$

$$= \nabla^2 p$$

i.e.

$$\nabla^2 p = \left(r \frac{\partial}{\partial r} + 1 \right) \left(r \frac{\partial p}{\partial r} \right) \qquad (8)$$

Thus even when vorticity is present, p satisfies a differential equation which is well known from usual steady conical flow problem. The problem is immediately solved by introducing the Chaplygin transformation, with boundary values determined by consideration of Rankine-Hugoniot conditions. Accordingly we have

$$\nabla_s^2 p = 0 \qquad (9)$$

where

$$s = \frac{1 - \sqrt{1-r^2}}{r} \tag{10}$$

is the Chaplygin transformation. This is the approach by which
Lighthill[2] calculated the pressure distribution.

(ii) Density perturbation, since

$$\nabla \rho = M_1 \nabla p \tag{11}$$

so in a region where there is no vorticity, it is immediately obtained
that $\rho = M_1 p$. However, it should be noted here that C_o is now
divided into two separate regions by the slip stream, and this nice
simple relation holds only in region OTP. For region OTQ, ρ has
different boundary condition than p and the solution will accordingly
be different. This fact is clearly born out by Ludloff's calculation.[3]

(iii) Velocity perturbation

or
$$\underline{r} \cdot \underline{r} \cdot \nabla \vec{q} = \nabla \cdot \vec{q} \quad , \quad \nabla \times \underline{r} \cdot \nabla \vec{q} = \nabla \times \nabla p = 0$$

$$(\underline{r} \cdot \nabla + 1)(\nabla \times \vec{q}) = 0$$

By elimination, it is easily obtained that

$$(\underline{r} \cdot \nabla + 2)\left[(\underline{r} \cdot \nabla + 1)(\underline{r} \cdot \nabla) - \nabla^2\right]\vec{q} = 0 \tag{12}$$

Now
$$(\underline{r} \cdot \nabla + 2)S = 0$$

gives
$$S = \frac{1}{r^2}\varphi(\theta)$$

So on introducing Chaplygin transformation (10), we obtain in general

$$\nabla_s^2 \vec{q} = f(s)\varphi(\theta) \tag{13}$$

i.e., u, v satisfies a Poisson differential equation.

However, if the flow is irrotational, elimination again results in equation (9).

$$\nabla^2 q = (\underline{r} \cdot \nabla + 1)(\underline{r} \cdot \nabla) q \qquad (13')$$

or

$$\nabla_s^2 q = 0$$

Thus, if we confine our attention to the vorticity *free* region OTP, it is seen that the problem simplifies to solving the equation (13') subject to following boundary conditions:

i) along wall OP, $v = 0$, $\dfrac{\partial u}{\partial q} = 0$; ii) over $\widehat{PT}$, i.e., $r = 1$, $u = v = 0$; and iii) on OT, normal velocity component vanishes since this is a slipstream.

(d) Higher Approximations

From above, it is clear that with boundary conditions properly determined, all first order quantities u, v, p, ρ can be calculated. Now suppose these have been found. A second approximation can be readily carried out in following manner:

Putting

$$S = S^{(1)} + S^{(2)} + S^{(3)} + \qquad (14)$$

where S denotes any of u, v, p, ρ, and collecting the second order terms in eqs. (6), we obtain

$$\underline{r} \cdot \nabla \rho^{(2)} - M_1 \, q^{(1)} \cdot \nabla \rho^{(1)} = M_1 [\rho^{(1)} \nabla \cdot q^{(1)} + \nabla \cdot q^{(2)}]$$

$$\underline{r} \cdot \nabla q^{(2)} - M_1 \, q^{(1)} \cdot \nabla q^{(1)} = -\rho^{(1)} \nabla p^{(1)} + \nabla p^{(2)} \qquad (15)$$

$$\underline{r} \cdot \nabla p^{(2)} - M_1 \, q^{(1)} \cdot \nabla p^{(1)} = \gamma M_1 \, p^{(1)} \nabla \cdot q^{(1)} + \nabla \cdot q^{(2)}$$

Thus the equation for the second-order term in p becomes

$$\nabla^2 p^{(2)} - (r\frac{\partial}{\partial r} + 1)(r\frac{\partial p^{(1)}}{\partial r})$$

$$= M_1 \left\{ \begin{array}{l} p^{(1)}\nabla^2 p^{(1)} + |\nabla p^{(1)}|^2 - \vec{q}^{(1)} \cdot \nabla p^{(1)} \\ -[\vec{q}^{(1)} \cdot \nabla + (r\frac{\partial}{\partial r} + r)p^{(1)}][r\frac{\partial p^{(1)}}{\partial r}] \\ - 2J(\frac{u^{(1)}, v^{(1)}}{x, y}) \end{array} \right\} \qquad (16)$$

or $\quad \nabla_s^2 p^{(2)} = \mathcal{F}(p^{(1)}, \vec{q}^{(1)})$

The corresponding boundary conditions over the wall can be obtained by standard procedure, those along Mach and reflected shocks should be obtained by first determining the second order values on the original boundary using Lighthill's technique. Thus the work involved in a second order solution will in general be prohibitive, and accordingly not attempted. Only the shock strength of reflected wave, which is of physical interest, is investigated in next part of this report.

II Strength of Reflected Shock

Since for small wall deflection angle ϑ the flow behind straight shock and slip stream is irrotational,* we are justified to introduce a velocity potential Φ , which, on account of the conical property shown before, further reduces to the form

$$\Phi(r^*, \theta, t) = a_1^2\, t\, f(r, \theta) \qquad (1)$$

where we have put $\quad a_1 \equiv \dfrac{\sigma p_1}{\rho_1}\ , \qquad r = \dfrac{r^*}{a_1 t}\ .$

Now denoting partial derivatives by subscripts, the radial and tangential velocity components u , v can be written

$$u = \Phi_{r^*} = f_r\, a_1$$
$$v = \frac{1}{r^*}\Phi_\theta = \frac{a_1}{r}\, f_\theta$$

and since

$$(\Phi_t)_{r^*} = (\Phi_t)_r + \Phi_r\, r_t = (f - r f_r)\, a_1^2$$
$$(\Phi_{tt})_{r^*} = \frac{r^2}{t}\, f_{rr}\, a_1^2$$

So we have

i)
$$\Phi_t + \frac{1}{2}(u^2 + v^2) + \frac{a^2}{\sigma - 1} = \frac{a_1^2}{\sigma - 1}$$

$$a^2 = \left\{1 - (\sigma-1)\left[f - r f_r + \frac{1}{2}(f_r^2 + r^{-2} f_\theta^2)\right]\right\} a_1^2 \qquad (2)$$

ii)
$$a^2 \nabla_{r^*}^2 \Phi = \Phi_{tt} + 2\sum u_i \frac{\partial u_i}{\partial t} + \sum\sum u_i u_j \frac{\partial u_i}{\partial x_j}$$

$$\frac{a^2}{t}\nabla^2 f = a_1^2\left[\frac{r^2}{t} f_{rr} + 2\left[f_r f_{rr}\left(-\frac{r}{t}\right) + \frac{1}{tr} f_\theta\left(\frac{1}{r}f_\theta - r f_{r\theta}\right)\right] \right.$$
$$\left. + \frac{1}{t}\left[-f_r^2 f_{rr} + \frac{f_\theta^2}{r^4} f_{\theta\theta} + \frac{f_r f_\theta}{r}\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}f_\theta\right) + \frac{1}{r\partial\theta} f_r\right\}\right]\right]$$

*$\overline{\quad\Omega \lesssim O(\vartheta^3)\quad}$ Bargmann, p. 16

as
$$\nabla^2_{r*} f = \frac{1}{t} \nabla^2 f$$
$$u_t = -\frac{a_1 r}{t} f_{rr}$$
$$v_t = -\frac{a_1 r}{t} \left(-\frac{f_\theta}{r^2} + \frac{1}{r} f_{r\theta}\right)$$

i.e.
$$\frac{a^2}{a_1^2} \nabla^2 f = r^2 f_{rr} + 2\left[-r f_r f_{rr} + \frac{1}{r^2} f_\theta^2 - \frac{1}{r} f_\theta f_{r\theta}\right]$$
$$+ \left[f_r^2 f_{rr} + \frac{f_\theta^2}{r^4} f_{\theta\theta} - \frac{f_r f_\theta^2}{r^3} + \frac{2}{r^2} f_r f_\theta f_{r\theta}\right] \qquad (3)$$

Eliminating $\frac{a}{a_1}$ from i) and ii)

$$\left[f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta}\right]\left\{1 - (r-1)\left[f - r f_r + \frac{1}{2}\left(f_r^2 + r^{-2} f_\theta^2\right)\right]\right\}$$

$$= r^2 f_{rr} - 2 r f_r f_{rr} + \frac{2}{r^2} f_\theta^2 - \frac{2}{r} f_\theta f_{r\theta}$$

$$+ f_r^2 f_{rr} + \frac{f_\theta^2}{r^4} f_{\theta\theta} - \frac{f_r f_\theta^2}{r^3} + \frac{2}{r^2} f_r f_\theta f_{r\theta}$$

So
$$\left\{1 - r^2 - (r-1)\left[f + \frac{1}{2} \frac{1}{r^2} f_\theta^2\right] + (\sigma+1)\left[r f_r - \frac{1}{2} f_r^2\right]\right\} f_{rr}$$

$$+ \left[\text{terms involving } f_r, f_{r\theta}, f_{\theta\theta}\right] = 0$$

$$\qquad (4)$$

Referring to coordinates fixed with respect to shock, the Rankine-Hugoniot condition can be written

i) $q^{(t)}$ = continuous $\qquad$ (5)

ii) $\Delta q^{(n)} = \frac{2}{r+1}\left(\frac{a^2}{q^{(n)}} - q^{(n)}\right)$

where $q^{(t)}$ and $q^{(n)}$ denote velocity components tangential and normal to the shock respectively, and $\Delta(\ )$ denotes the increase of a quantity across the shock front.

Now since ϕ exists, and flow is steady in the (r, θ) plane, let the reflected shock front be $r = \eta(\theta)$, then

$$u = a_1 (-\eta + f_r) \qquad v = \frac{a_1}{r} f_\theta$$

$$q^{(t)} = \frac{f_\theta + \eta'(\theta)(f_r - \eta)}{r\sqrt{1 + r^{-2}\eta'^2}} \, a_1$$

$$q^{(n)} = \frac{-(f_r - \eta) + r^{-2}\eta' f_\theta}{\sqrt{1 + r^{-2}\eta'^2}} \, a_1$$

i) gives $\quad \Delta\left[f_\theta + \eta'(\theta)(f_r - \eta) \right] = 0$

ii) $\quad \Delta q^{(n)} = \dfrac{-\Delta f_r - r^{-2}\eta'^2 \Delta f_r}{\sqrt{1 + r^{-2}\eta'^2}} \, a_1 = -\sqrt{1 + r^{-2}\eta'^2} \, a_1 \, \Delta f_r$

i.e., $\quad \dfrac{2}{\sigma^2 + 1}\left(\dfrac{a^2}{q^{(n)}} - q^{(n)} \right) = \dfrac{2}{\gamma + 1}\left[\dfrac{1 - (\gamma - 1)(f - r f_r)}{q^{(n)}/a_1^2} - q^{(n)} \right]$

Therefore we have

$$\Delta f_r = -\frac{2}{\sigma^2 + 1}\left[\frac{a^2}{a_1^2(\eta - f_r + r^{-2}\eta' f_\theta)} - \frac{\eta - f_r + r^{-2}\eta' f_\theta}{1 + r^{-2}\eta'^2} \right]$$

$$= -\frac{2}{\gamma + 1}\left[\frac{1 - (\gamma - 1)(f - r f_r)}{1 + (\eta - 1) - f_r} - [1 + (\eta - 1) - f_r] \right] \qquad (6)$$

$$= \frac{2}{\sigma^2 + 1}\left[(\sigma^2 - 1)(f - r f_r) - 2 f_r + 2(\eta - 1) \right]$$

Since $f_1 = f_{1R} = 0$ on $R = 1$ and $f_{1R} \sim A(\theta)\sqrt{1 - R}$ in linear theory.

But

$$\frac{p - p_1}{\rho_1 \sigma^2} = -\frac{\phi_t}{\sigma^2} = \frac{-f + r f_r}{M_1^2} \sim A(\theta)\sqrt{1-R} \qquad R \to 1 \qquad (7)$$

So

$$f_{1R} = M_1^2 A(\theta)\sqrt{1-R} \qquad (8)$$

Following Lighthill [4] by putting

$$r = R + r(\theta) = R + r_1(\theta) + r_2(\theta) + \cdots \qquad (9)$$

$$f = f_1(R,\theta) + f_2(R,\theta) + \cdots$$

in (4) and condition on $R = 1$, that the linear coefficients in (4) must vanish; and noting that although since

$$\frac{\partial}{\partial r} \to \frac{\partial}{\partial R} \qquad \frac{\partial}{\partial \theta} \to \frac{\partial}{\partial \theta} - r'(\theta)\frac{\partial}{\partial R}$$

terms $f_{r\theta}$ and $f_{\theta\theta}$ may produce terms in f_{RR}, they are however all not of dimension 1; we have

$$-2 R r_1 - (\gamma-1)(f_1 - R f_{1R}) + 2 R f_{1R} = 0 \qquad \text{on} \quad R = 1 \qquad (10)$$

i.e. $r_1(\theta) \equiv 0$

Now

$$\lim_{R \to 1} \left[(\gamma-1) R f_{1R} + 2 R f_{1R} \right] f_{1RR} = (\gamma+1) R f_{1R} f_{1RR}$$

$$= -\frac{\gamma+1}{2} M_1^4 A^2(\theta)$$

So

$$f_{2R} + f_{2\theta\theta} = \frac{\gamma+1}{2} M_1^4 A^2(\theta)$$

$$\left[-2 r_2 - (\gamma-1)(f_2 - f_{2R}) + 2 f_{2R} \right]_{R=1} = 0 \qquad (11)$$

$$r_2(\theta) \sim A^2(\theta)$$

The shock condition at $r = \eta(\theta)$ are: Since $f \equiv 0$ ahead of shock, so directly behind it

$$f = 0 \qquad f_R = \frac{4(\gamma-1)}{\gamma+1} \qquad (12)$$

Shock cones in range $R < 1$, $\eta < 1 + r_2 + r_3 + \cdots$ so $\eta - 1$ of dimension 2, $\eta = 1 + \eta_2 + \eta_2 + \cdots$, $0 \le \eta_2 \le r_2$.

In condition $f = 0$ on $r = \eta(\theta)$, terms f_1, f_2,...can be expanded by Taylor expansion (at R = 1) using $R = 1 + (\eta_2 - r_2) + (\eta_3 - r_3) + \cdots$

Terms of dimension 1 and 2 give

$$(f_1)_{R=1} = 0 \qquad (f_2)_{R=1} = 0$$
$$(f_{2R})_{R=1} = \frac{\gamma+1}{2} M_1^4 A^2(\theta) \qquad (13)$$
$$r_2(\theta) = \frac{(\gamma+1)^2}{4} M_1^4 A^2(\theta)$$

To express the second condition of (12), f_R is needed when

$$R = 1 + (\eta_2 - r_2) + (\eta_3 - r_3) + \cdots$$

The largest terms here are of dimension 2, and these come from

$$f_{1R} \sim M_1^2 A(\theta) \sqrt{1-R} \qquad \text{and from} \qquad (f_{2R})_{R=1}$$

so they are

$$M_1^2 A(\theta) \sqrt{r_2 - \eta_2} + (f_{2R})_{R=1}$$

by (12) and (13)

$$M_1^2 A(\theta) \sqrt{\frac{(\gamma+1)^2}{4} M_1^4 A^2(\theta) - \eta_2} + \frac{\gamma+1}{2} M_1^4 A^2(\theta) = \frac{4\eta_2}{\gamma+1} \qquad (16)$$

Putting $\eta_2 = \frac{(\gamma+1)^2}{4} M_1^4 K(\theta)$, this becomes

$$A(\theta) \sqrt{A^2(\theta) - K(\theta)} = 2K(\theta) - A^2(\theta) \qquad (15)$$

The solution of (15) gives

$$K(\theta) = \frac{3}{4} A^2(\theta) \qquad A(\theta) > 0$$
$$= 0 \qquad A(\theta) < 0 \qquad (16)$$

For the case of compression, $A(\theta) > 0$, the shock is thus located at

$$r = 1 + \frac{3}{16} (\gamma+1)^2 M_1^4 A^2(\theta) \qquad (17)$$

Now the pressure change across the shock is

$$\rho_1 \phi_t = \frac{\rho_1 U^2}{M_1^2} (r f_r - f) = \rho_1 U^2 \frac{4(\eta-1)}{(\gamma+1) M_1^2}$$

Since $\gamma p_1 = a_1^2 \rho_1$ the shock strength is given by

$$\frac{\Delta p}{p_1} = \frac{4 \gamma U^2}{(\gamma+1) M_1^2 a_1^2} \eta_2(\theta) = \frac{3}{4} \gamma (\gamma+1) M_1^4 A^2(\theta) \qquad (18)$$

where $A(\theta) = \lim_{r \to 1} \left(\frac{p - p_1}{\rho_1 U^2 \sqrt{1-r}} \right)$

on linear theory

$$= \frac{\partial p}{\partial n} \qquad \text{in Chaplygin plane.}$$

and $\frac{\partial p}{\partial n}$ as given by Ref. 2 (59) is

$$\frac{\partial p}{\partial n} = \left| \frac{dz_1}{ds} \right| \frac{c[D(x_1 - x_0) - 1]}{-(x_1^2 - 1)^{\frac{1}{2}}(x_1 - x_0)[\alpha + \sqrt{1-x_1}][\beta + \sqrt{1-x_1}]} \quad \nu$$

$\frac{\partial p}{\partial n}$ at triple point $(x = R, \; x_1 = \infty)$ is 0 , so strength of reflected shock vanishes at the triple point.

CONCLUSIONS

The conical property in time-space of diffraction of propagating plane shock is discussed. Second order solution for pressure p has been suggested, but since this requires more information regarding u , v and the second-order boundary conditions than is presently available, actual carrying out of the solution is not attempted.

The strength of reflected shock, however, has been investigated; and it is found to be generally of order β^2 . In particular, this vanishes at the triple point. The immediate conclusion is that while the strength is of second order in general, it can be at most of third order at triple point in particular.

BIBLIOGRAPHY

1. Bargmann, V. "On Nearly Glancing Reflection of Shocks" AMP Report 108, 2R, 1945.

2. Lighthill, M. J. "The Diffraction of Blast, I" Proc. Roy. Soc. A, 198, 454-470, 1949.

3. Ting, L. and Ludloff, H. F. "Aerodynamics of Blasts," Readers Forum, J.A.S. Feb. 1951.

4. Lighthill, M. J. "The Shock Strength in Supersonic Conical Fields" Phil. Mag. Ser. 7, Vol. XI, p. 1202, Dec. 1949.

APPENDIX

The Reflected-Shock Angle at the Triple Point

A slight extension of this investigation discloses
that at the triple point where the incident, reflected and
Mach shocks meet, not only is the reflected shock of vanishing
strength but also is it locally tangent to the acoustic circle.
The latter statement can be proved as follows:

Denoting by ψ the angle included between the
reflected shock and acoustic circle, then

$$\mathrm{tg}\,\psi = \frac{dr}{r\,d\theta} \tag{19}$$

where r is given by Eq. (17).

Now the mapping from the $S = \rho\,e^{i\theta}$ plane
to the $z_1 = x_1 + i\,y_1$ plane, in Lighthill's paper[2], is
everywhere analytic, so $\left|\frac{dz_1}{dS}\right|$ and its derivatives are
finite. Even though the mapping from the physical conical
plane $r\,e^{i\theta}$ to the S plane is singular on the unit
circle, the following relations hold:

$$\frac{\partial}{\rho\,\partial\theta} = \frac{\partial}{r\,\partial\theta} \qquad \frac{\partial}{\partial r} = \frac{\partial}{\partial\rho}\,\frac{1}{\sqrt{1-r^2}} \tag{20}$$

So it is immediately seen that this singular behavior will not
affect the determination of ψ . Furthermore, since

$$\frac{\partial}{\rho\,\partial\theta} = \frac{\partial}{\partial t} = \frac{\partial}{\partial x}\left|\frac{dz_1}{dS}\right|$$

we eventually get

$$\mathrm{tg}\,\psi = \frac{\partial r}{\rho\,\partial\theta} = \frac{3}{16}\,(\sigma+1)^2 M_1^4\,\vartheta^2\left|\frac{dz_1}{dS}\right|\frac{\partial A^2}{\partial x} \tag{21}$$

and the conclusion follows at once.